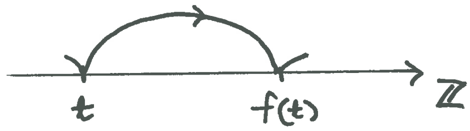
Mathematics of Juggling (5cr)

- Webpage: math.aalto.fi/~havarpan/juggling
  Noppa will not be used
- Introduction / motivation: live only

I Periodic siteswaps

(SEE EXAMPLE ON PAGE 3.)

- Any function $f: \mathbb{Z} \to \mathbb{Z}$ such that
  a) f is bijective
  b) $\forall t \in \mathbb{Z}: f(t) \geq t$



can be interpreted as a juggling pattern:

- $\mathbb{Z}$ beats (eternal pattern)
  ~~one hand (at a time)~~
- a particle thrown at $t \in \mathbb{Z}$ is next thrown at $f(t)$
- if $f(t) = t$, the juggler waits (hand empty at t).

Practical ~~matters:~~ restrictions:

(*)
- one hand (at a time)
- at most one particle thrown at a time ("particles don't collide") CONDITION a)
- particles are thrown forward in time. b)

(*): All these three simplifications can be dropped. (We will drop b) at some point.)

Moreover:

- throws are immediate ("dwell time" neglected) I.E. THE TIME THE PARTICLE SPENDS IN THE HAND
- spatiality neglected (model temporal only).

- Functions $f: \mathbb{Z} \to \mathbb{Z}$ satisfying a) & b) are called <u>siteswap juggling patterns</u> or just <u>siteswaps</u>. REASON: EXRC. 1 / PROB. 1.

- Siteswaps form a subset of bijections $\mathbb{Z} \to \mathbb{Z}$, i.e. permutations of the integers.

- The <u>number of particles</u> in a siteswap f equals the number of ITS nontrivial orbits

$$f^k = \begin{cases} \underbrace{f \circ f \circ \ldots \circ f}_{k \text{ TIMES}}, & k \geq 1 \\ (f^{-1})^{-k}, & k \leq -1 \\ \mathrm{id}, & k = 0 \end{cases}$$

$$\underbrace{\{ f^k(t) \mid k \in \mathbb{Z} \}}_{\text{All the throw times of a particle thrown at } t \text{ (fixed)}} \neq \{t\}.$$

- The <u>throw</u> at time t in f is

$$f(t) - t =: d(t),$$ ZERO IF EMPTY THROW

and ~~often~~ f is <u>n-periodic</u> $(n \geq 1)$ if

$$d(t+n) = d(t) \quad \forall t \in \mathbb{Z}.$$

- Periodic siteswaps are usually given by listing the throws over one period, e.g. $\underbrace{423}_{= (4,2,3)}$, 51, 55050.

EQUIVALENT TO $(2,3,4)$ & $(3,4,2)$.

- **Theorem.** A sequence $a_1, \dots, a_n \in \mathbb{N}$ represents[⊛] a n-periodic siteswap iff

$$\{(a_i + i) \bmod n \; : \; i = 1, \dots, n\} = \{0, \dots, n-1\}.$$

⊛ : $a_i = d(i)$

Proof: Exrc. 1 / Prob. 2. $\square$

- **Theorem.** The number of particles in a[⊛] siteswap $a_1, \dots, a_n$ equals

⊛ PERIODIC

$$\frac{1}{n} \sum_{i=1}^{n} a_i .$$

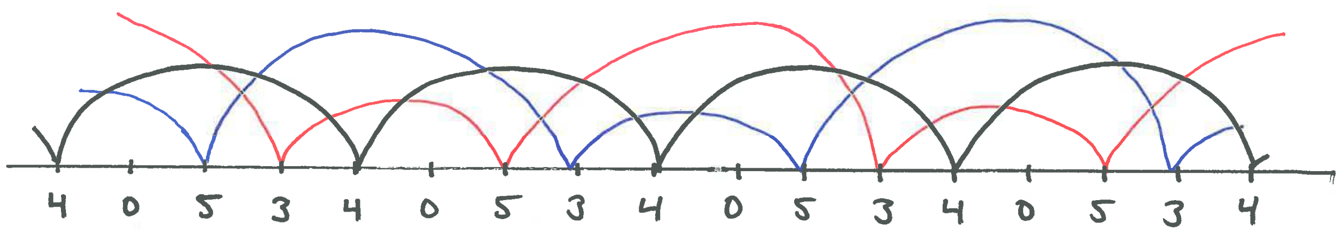
Proof: Exrc. 1 / Prob. 3. $\square$

- **Example.** The seq. 4053 is a siteswap:

$$\begin{array}{rcccc} & 4 & 0 & 5 & 3 \\ +_4 & 0 & 1 & 2 & 3 \\ \hline & 0 & 1 & 3 & 2 \end{array} \leftarrow \{0,1,2,3\} \text{ OK!}$$

with $\frac{4+0+5+3}{4} = 3$ particles.

Let us draw a diagram of the pattern:



Particles:

40004000
00500003
00030050

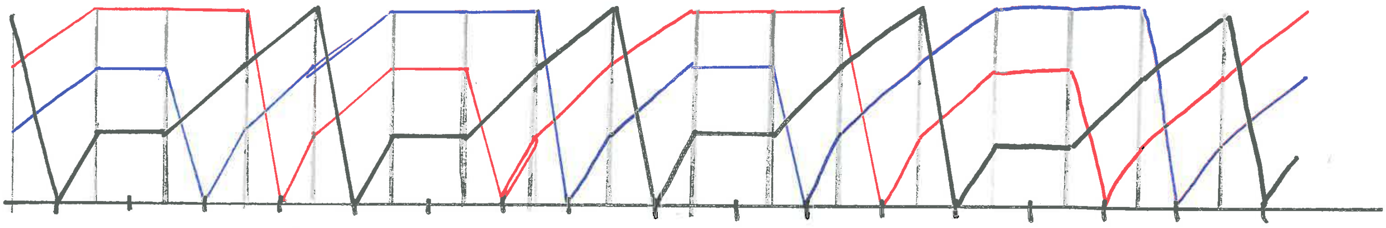
} (three ~~different~~ separate one-particle patterns).

JUGGLING CARDS:

Let us ~~them~~ ABOVE reshape the orbits and add vertical bars:

• <u>Juggling cards.</u> (Example continues.)

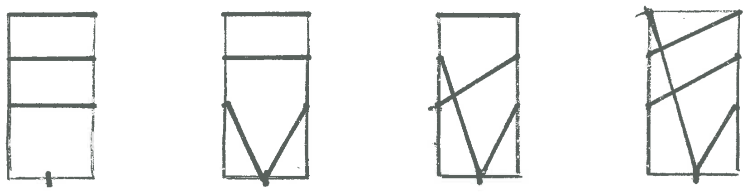
In the previous diagram, let us ~~set~~ reshape the orbits and add vertical bars:



4 0 5 3 4 0 5 3 4 0 5 3 4 0 5 3 4

The new diagram can be obtained by placing "cards" side by side.

The <u>3-particle juggling cards</u> are



C_0 C_1 C_2 C_3 ,

and the <u>card sequence</u> for 4053 (= ...40534053...) is $C_3 C_0 C_2 C_3$ $(= \ldots C_3 C_0 C_2 C_3 C_3 C_0 C_2 C_3 \ldots)$.

In general, siteswaps $\cong$ card sequences ($\cong$ means 'isomorphic' or 'one-to-one correspondence'), and we obtain

• <u>Theorem.</u> The number of n-periodic siteswaps with at most b particles equals $(b+1)^n$, when cyclic shifts are counted as distinct. (E.G. 234, 423) $\square$

MORE: EXRC. 1 / PROB. 4.

II Siteswap state graphs

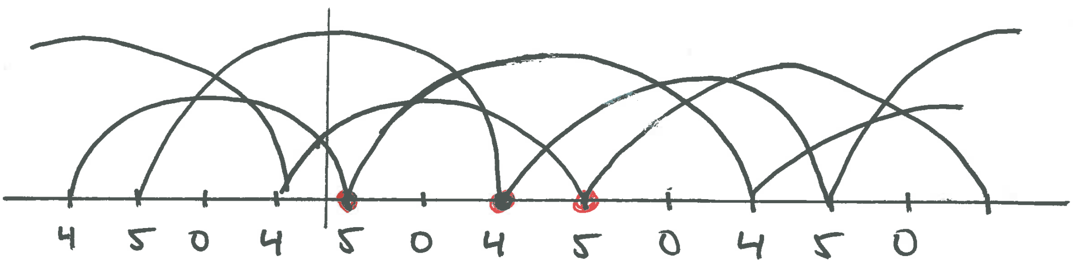
- Let $f : \mathbb{Z} \to \mathbb{Z}$ be a siteswap (not necessarily periodic), and fix $t \in \mathbb{Z}$. Define the *state of f at t* as

$$\cancel{\{ i \geq t \mid f^{-1}(i) < t \}}$$

$$\{ i - t \mid i \geq t \text{ and } f^{-1}(i) < t \},$$

i.e. the scheduled future throw times, relative to t, of the particles thrown before t. (PARTICLES "IN THE AIR".)

- Example. 450 at any 5-throw:



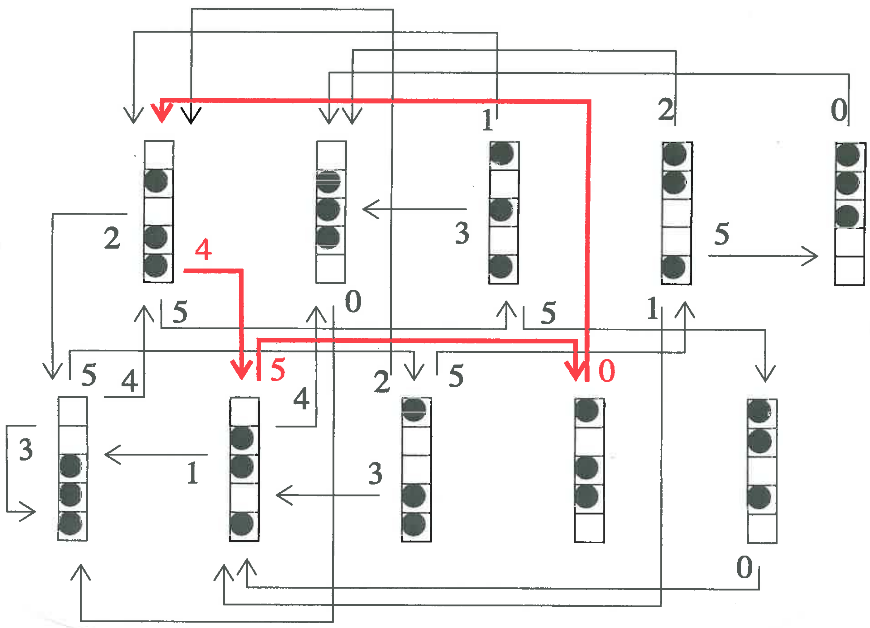
Here the state is $\{0, 2, 3\}$; let's write it as $\underbrace{\overset{0}{\times} - \overset{2}{\times}\overset{3}{\times}}_{\text{BEATS FROM "NOW"}} \underbrace{- - - \cdots}_{\text{NO NEED TO WRITE THIS PART}}$ or just

x - xx .

$\therefore$ The state cycle of 450:

$$\begin{tikzcd} \text{x-xx} \arrow[rr, "5"] & & \text{-xx-x} \arrow[dl, "0"] \\ & \text{xx-x} \arrow[ul, "4"] & \end{tikzcd}$$

• Example. Any siteswap (not necess. periodic) with 3 particles and max. throw ≤ 5 is a path in the following graph (from the poster; states drawn vertically):



• The b-particle ground state is $\{0,\dots,b-1\}$, and any periodic siteswap that contains it in the state cycle is a ground state pattern.
A pattern that is not ground state is excited state (such as 450 above).

- The state dynamics. (A PRECISE DEFINITION OF THE STATE GRAPH.)

Any siteswap with b particles and max. throw $\le h$ is a path in the (b,h) state graph: (DIRECTED.)

- states: sets $A \subset \{0,\dots,h-1\}$ such that $|A| = b$

HERE $|A| = \#A$, THE SIZE OF A. THERE ARE $\binom{h}{b}$ STATES.

- Given (b,h)-states A and B, B is a follower of A if either

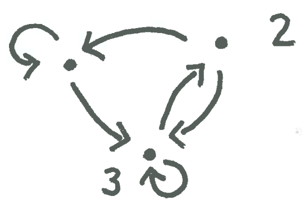
1) $0 \notin A$ and $B = A-1$ $= \{x-1 \mid x \in A\}$

or

2) $0 \in A$ and $B = (A^* - 1) \cup \{x\}$ for some $x \in \{0,\dots,h-1\} \setminus (A^* - 1)$, where $A^* = A \setminus \{0\}$.

- The transition matrix in a state graph.

Consider the following (general/abstract) state graph (= simple directed graph with loops allowed) with states 1, 2 and 3:



Define its <u>transition matrix</u> as a 3×3 matrix $T = (T_{ij})$:

$$T_{ij} = \begin{cases} 1, & \text{if } j \text{ is a follower of } i \\ 0 & \text{otherwise}. \end{cases}$$

(THE DEFINITION IS GENERAL.)

That is,

$$T = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

In general we have

<u>Theorem</u>. Let G be a state graph with n states, and let $T \in \{0,1\}^{n\times n}$ be its transition matrix. Then the number of k-step paths $(k \geq 1)$ from state i to state j equals $(T^k)_{ij}$.

<u>Proof</u>. Exrc. 1 / Prob. 5.