Tue Apr 21

III Affine permutations.

- Recall: the symmetric group

$$S_n = \{ \text{bijections } \{1,\dots,n\} \to \{1,\dots,n\} \}$$

is generated by simple transpositions

$$s_i = (i \;\; i+1), \quad i = 1,\dots,n-1.$$

(CYCLE NOTATION.)

"MATRIX" NOTATION:

$$s_i = \begin{pmatrix} 1 \cdots i & i+1 & \cdots n \\ 1 \cdots i+1 & i & \cdots n \end{pmatrix}.$$

(This means that every permutation is a product of transpositions.) (UNDER FUNCTION COMPOSITION.)

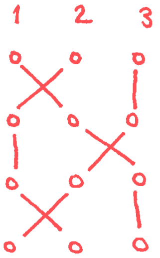
The group S_n can be thought of as a <u>reflection group</u> in $\mathbb{R}^n$: one permutes the subscripts of the standard basis vectors $\bar{e}_1,\dots,\bar{e}_n$. A transposition $(i \; j)$ sends $\bar{e}_i - \bar{e}_j$ to its negative, while fixing pointwise its orthogonal complement (the "mirror").

One often writes the relations for the generators:

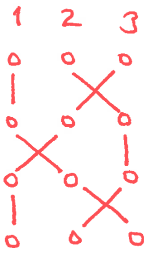
$$s_i^2 = 1 \quad (= \mathrm{id}),$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \quad \text{"BRAID RELATION"}$$

$$|i-j| \geq 2 \Rightarrow s_i s_j = s_j s_i.$$



$s_1 s_2 s_1$



$s_2 s_1 s_2$

"HOW TO CALCULATE IN S_n"

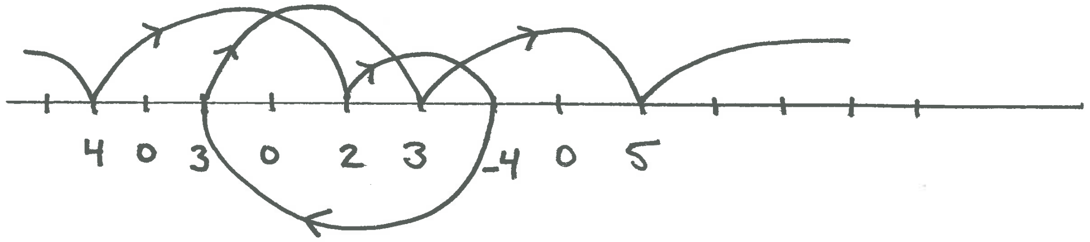
In the reflection context above, S_n is called A_{n-1} (Weyl / Coxeter / Reflection group of type A.) (EXPLAIN SUBSCRIPT $n-1$.)

- The <u>affine S_n</u> is defined as

$$\tilde{S}_n = \left\{ \text{bijections } f: \mathbb{Z} \to \mathbb{Z} ,\ f(t)+n = f(t+n) \ \overset{\forall t \in \mathbb{Z}}{} \right\},$$

i.e. n-periodic siteswaps without the restriction $f(t) \geq t$.

<u>Interpretation</u>: particles are (OR CAN BE) thrown also backwards in time, or equivalently, antiparticles are thrown forwards in time:



DOESN'T LOOK VERY PERIODIC. SORRY.

↑ CHANGE THE DIRECTION OF THIS ARROW TO OBTAIN THE PARTICLE/ANTIPARTICLE INTERPRETATION: AT THE -4 THROW, AN ANTIPARTICLE CREATED 4 BEATS AGO ANNIHILATES WHEN HITTING A PARTICLE. AT THE FIRST 3-THROW, A PARTICLE/ANTIPARTICLE PAIR IS CREATED OUT OF NOTHING.

If one allows negative throws a_i, the "permutation test" stays the same:

$$\{(a_i + i) \bmod n\} = \{0, \dots, n-1\},$$

and $\frac{1}{n}\sum_{i=1}^{n} a_i$ now gives the "total mass", i.e.

(# of particles) − (# of antiparticles).

BY SUBTRACTING THE TOTAL MASS FROM THE THROWS,

- Any siteswap can be decomposed as a zero-ball pattern and a constant-throw pattern.

(ZERO-BALL OR ZERO-PARTICLE.)

- Example

$$423 \underset{\substack{\uparrow \\ \text{FOR CLARITY}}}{=} (4,2,3) \underset{\substack{\uparrow \\ \text{SUBTRACT 3} \\ \text{FROM ALL}}}{=} \underbrace{(1,-1,0)}_{\substack{\text{ZERO-BALL} \\ \text{PATTERN,} \\ \text{"ROTATION PART"}}} \oplus \underbrace{(3,3,3)}_{\substack{\text{CONSTANT-THROW} \\ \text{PATTERN,} \\ \text{"TRANSLATION PART"}}} .$$

- The space of zero-ball patterns with period n, i.e.

$$\left\{ f \in \tilde{S}_n \;\middle|\; \sum_{i=1}^{n} (f(i)-i) = 0 \right\},$$

is called $\tilde{A}_{n-1}$. Now the generators (simple transpositions) are $s_0, \dots, s_{n-1}$, where

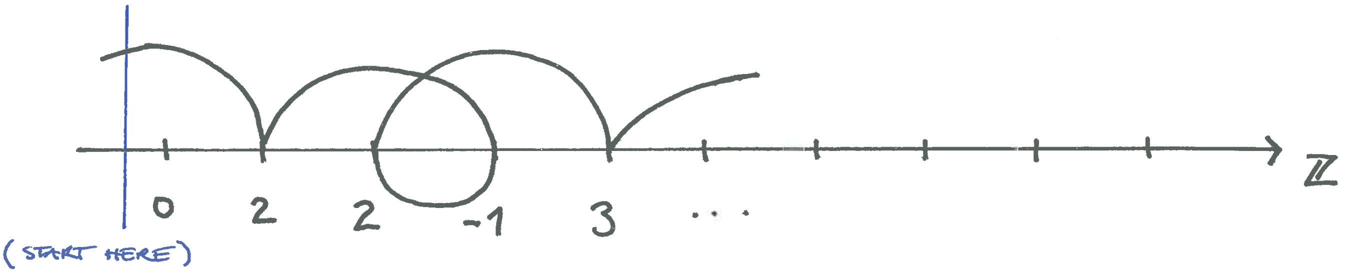
$$s_i(t) = \begin{cases} t+1, & t \equiv i \\ t-1, & t \equiv i+1 \\ t, & \text{otherwise} \end{cases} .$$

$\tilde{A}_{n-1}$ = AFFINE WEYL GROUP OF TYPE A.

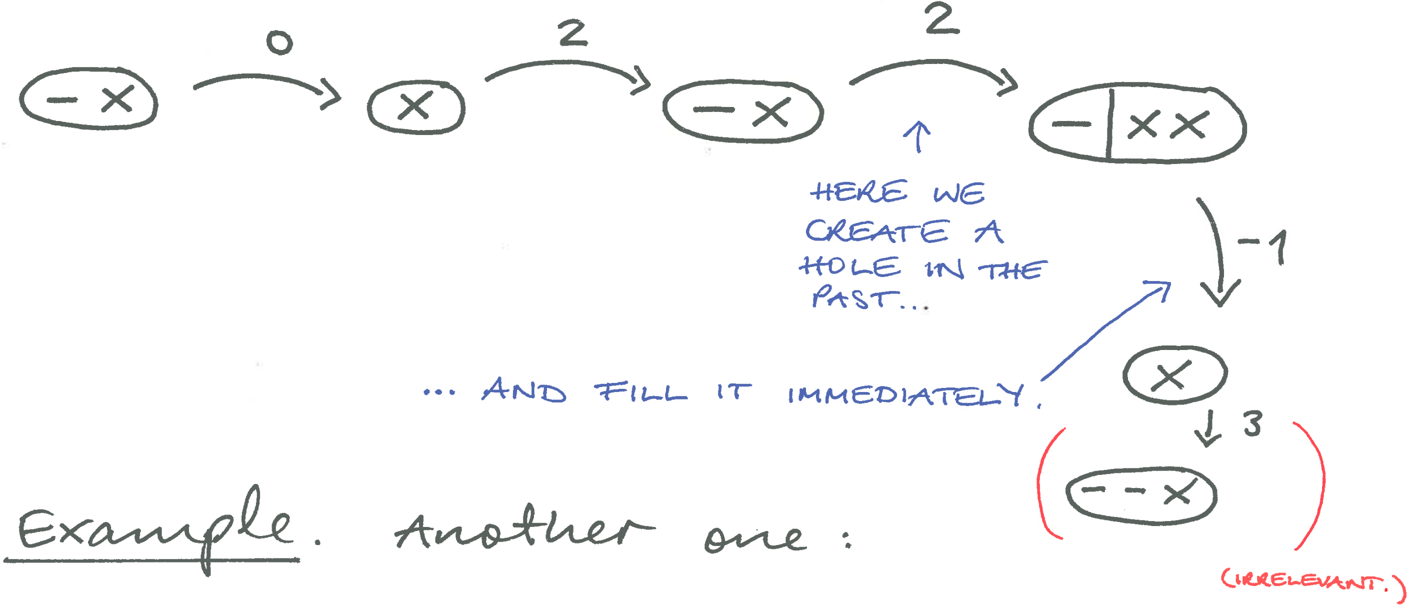
- siteswaps with negative throws allowed are called <u>virtual</u> siteswaps (or virtual juggling patterns).

• The virtual dynamics.

• **Example.** A one-ball pattern:



states:

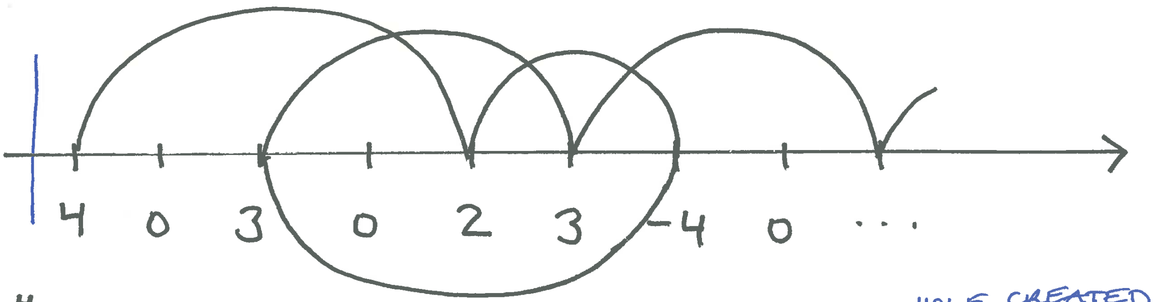


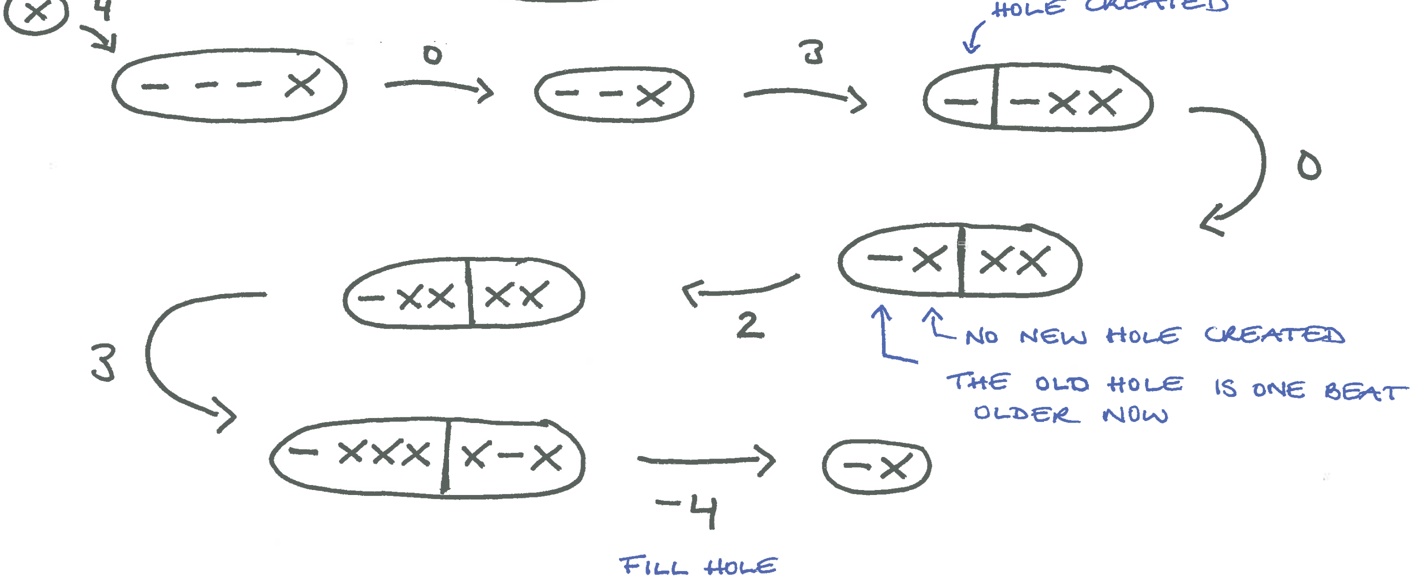
HERE WE CREATE A HOLE IN THE PAST...

... AND FILL IT IMMEDIATELY.

(IRRELEVANT.)

• **Example.** Another one:





HOLE CREATED

NO NEW HOLE CREATED

THE OLD HOLE IS ONE BEAT OLDER NOW

FILL HOLE

- A <u>virtual state</u> with $k \in \mathbb{Z}$ net particles is a set $A \subset \mathbb{Z}$ such that

$$|\mathbb{Z}_- \setminus A| + |\mathbb{Z}_+ \cap A| = k,$$

where

$$\mathbb{Z}_- = \{x \in \mathbb{Z} : x \leq -1\},$$

$$\mathbb{Z}_+ = \{x \in \mathbb{Z} : x \geq 0\},$$

and

$$\underbrace{|\mathbb{Z}_- \setminus A|}_{\text{HOLES IN THE PAST}} < \infty, \quad \underbrace{|\mathbb{Z}_+ \cap A|}_{\text{PARTICLES IN THE FUTURE}} < \infty.$$

- A state B <u>follows</u> a state A if $B = (A-1) \cup \{x\}$ for some $x \notin (A-1)$.

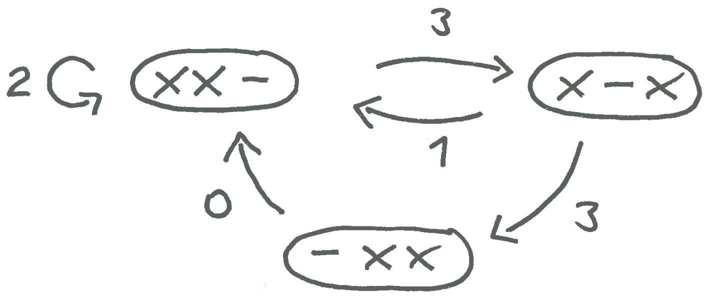
- <u>Example</u>. The zero-ball ground state is $\mathbb{Z}_-$ $\left(= \boxed{\cdots\cdot \times\times\times | - - - \cdots} \right)$,

and a non-virtual state A is a virtual state with $\mathbb{Z}_- \setminus A = \emptyset$.

- If we bound the throw numbers to an interval $[-h, h] \subset \mathbb{Z}$, then the k-particle state graph is finite and <u>isomorphic to the $(b, 2h)$ state graph</u>, where $b = k+h$.

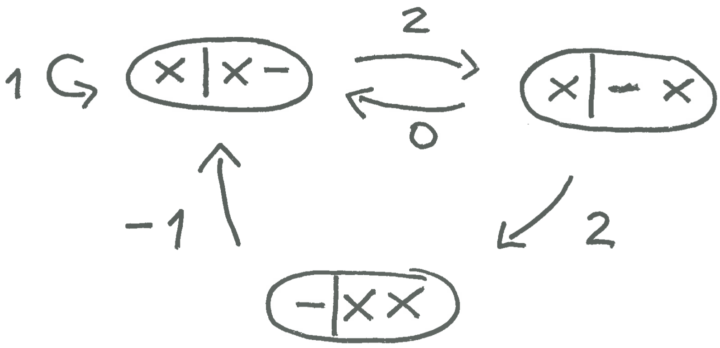
MOVE THE ORIGIN ONE STEP TO THE LEFT / RIGHT ⟺ ADD / SUBTRACT ONE TO EACH THROW NUMBER

- Example. The (2, 3) graph :



2 PARTICLES, THROWS ∈ [0, 3]

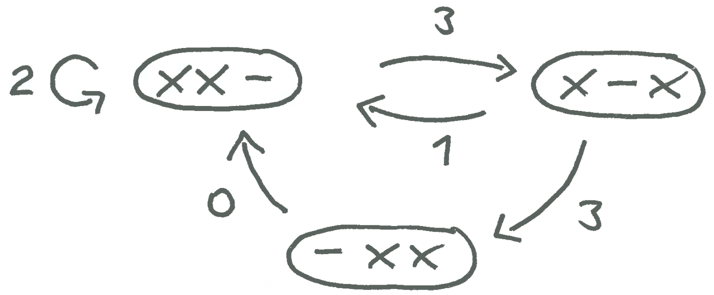
Subtract one from each throw & move the origin a step to the right :



1 PARTICLE, THROWS ∈ [−1, 2] .

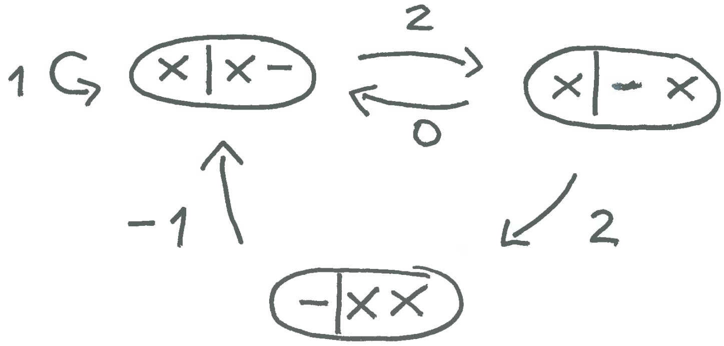
⟺ ADD / SUBTRACT ONE PARTICLE.

- Example: The (2,3) graph:



2 PARTICLES, THROWS ∈ [0, 3]

Subtract one from each throw & move the origin a step to the right:



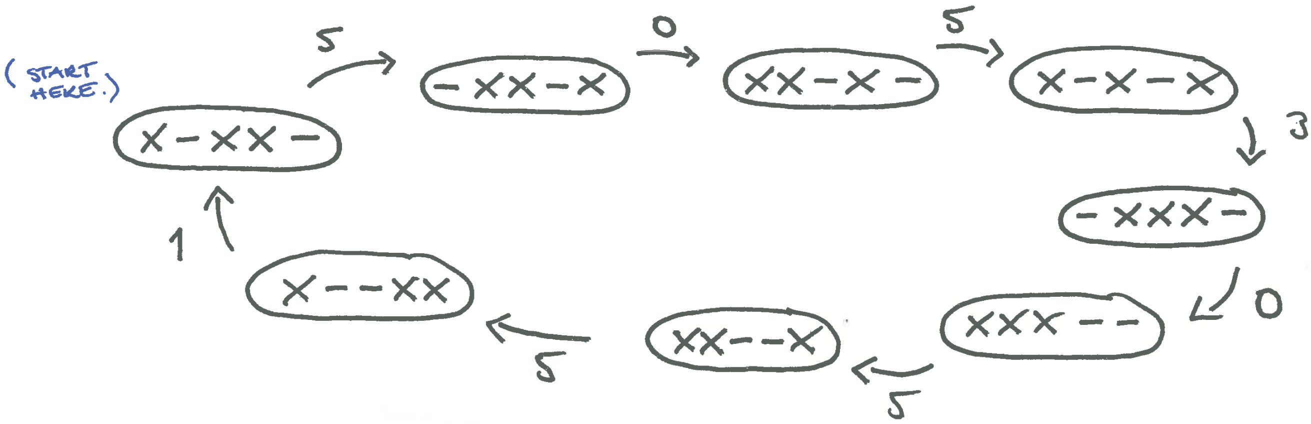
1 PARTICLE, THROWS ∈ [−1, 2].

Wed Apr 22

IV Prime patterns.

- A periodic siteswap $a_1 \cdots a_n$ (now non-virtual) is prime if the state cycle visits each of its state exactly once. (~~Equivalently: no~~ ~~siteswap~~ ~~repeatable subsequences in $a_1 \cdots a_n$.~~) ← (A BIT TOO VAGUE.)

- Example. 50530551 is prime:



It visits 8 of the 10 states in the (3,5) graph; only -x-xx and --xxx are missing.

(Example. 423 is not prime – see for yourself.)

- Open problem 1. (Hard.) Give a combinatorial expression (e.g. a recursion formula) for the number of prime patterns in the (b,h) graph.

- Open problem 2. (Perhaps easier.) Find the longest prime patterns in the (b,h) graph.

- Let us investigate prob. 2. Define a (b,h) shift cycle as a (prime) loop in the (b,h) graph that has only throws zero or h.

- **Example.** In the (3,5) graph there are two shift cycles: 55500 and 50505.

55500 ↑ CONTAINS $\boxed{\text{xxx--}}$

50505 ↑ CONTAINS $\boxed{\text{x-x-x}}$

- Any (b,h) graph is a union of distinct shift cycles ... I mean: the shift cycles cannot overlap, and any state belongs to one shift cycle. (shift cycles are equivalence classes in the set of states.)

- **Theorem.** The length of a prime pattern in a (b,h) graph is at most

$$(\#\text{ of states}) - (\#\text{ of shift cycles}).$$

For example, 50530551 is of maximum length, because there are 10 states and 2 shift cycles in the (3,5) graph.

Open problem 3. For which (b,h) is the maximum reached? Example: (3,5) YES
(3,6) NO.

Proof. Exercises 2 / Prob. 4. Start: consider a prime pattern that is not a shift cycle. Then it visits at least two shift cycles. Show that one state (at least) must be omitted from each shift cycle. (...) □

V Juggling Markov chains.

- A state graph is <u>irreducible</u> or <u>strongly connected</u> if it is possible to reach any state from any other state (with a finite # of steps).

- If T is the transition matrix, then irreducibility means that

$$\forall i,j \ \exists k : \ (T^k)_{ij} \geq 1 .$$

- The <u>period</u> of a state i in a state graph (matrix T) is

$$\gcd \left\{ k \geq 1 : (T^k)_{ii} \geq 1 \right\} .$$

- <u>Example.</u> If a state has period 2, then it is possible to travel from the state to itself with an even # of steps only. (I.e. odd # impossible.)

- A state graph is <u>aperiodic</u> if each state has period one.

- <u>Example.</u> In an aperiodic graph, you cannot say that the length of a path from a state to itself must be a multiple of some number (other than one).

- If T is the transition matrix, then irreducibility & aperiodicity means that

$\circledast$ $$\exists k \ \forall i,j : \ (T^k)_{ij} \geq 1 .$$

Proof.

- $\circledast \Rightarrow$ irred : clear.
- $\circledast \Rightarrow$ aper : Let $(T^k)_{ij} \geq 1 \ \forall i,j$.

  Then also $(T^{k+1})_{ij} \geq 1 \ \forall i,j$:

  Let m be the state just before j in a k-step path $i \rightarrow \dots \rightarrow m \rightarrow j$

  By assumption, $\exists$ k-step path $i \rightarrow \dots \rightarrow m$.

  Aperiodicity follows (as in Exrc. 2 / Prob. 5).
- Irred. & aper. $\Rightarrow \circledast$: Choose k large enough (the graph is finite).

- A vector $\overline{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$ is a *probability vector*, if

  $0 \leq x_i \leq 1 \quad \forall i \in \{1, \dots, n\}$ and

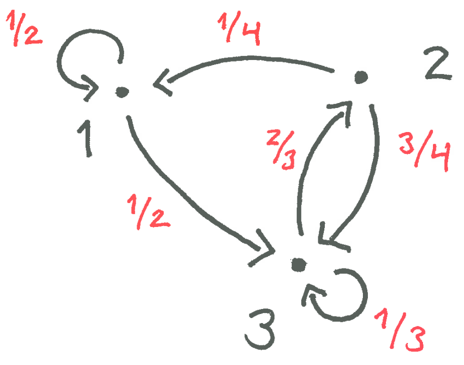
  $x_1 + \dots + x_n = 1$.

- ~~A matrix~~ An $n \times n$ matrix P is a *Markov matrix* (or probability matrix, or stochastic matrix) if its *rows* are probability vectors.

  ↑ THE STANDARD CONVENTION AMONG PROBABILISTS.

- Markov matrix ⟺ state graph with transition probabilities.

- *Example* our earlier example with "probability weights":



$$P = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 1/4 & 0 & 3/4 \\ 0 & 2/3 & 1/3 \end{pmatrix}$$

P_{ij} = PROBABILITY OF $i \to j$ (ONE STEP).