- Addendum. ($\to$ p. 22)

In any finite & irreducible state graph, all states have the same period.

Proof.

HIDDEN!

(Exercise.)

(we don't actually need this result.)

- Let us prove carefully: irred. & aper. & finite $\Rightarrow$ $\exists k : (T^k)_{ij} \geq 1 \; \forall i,j$. SEE p. 18.

First we need

- Lemma.

Let $A = \{a_i \mid i \geq 1\} \subset \mathbb{N}$ be such that

1) $a, b \in A \Rightarrow a + b \in A$

and

2) $\gcd(A) = 1$.

Then $\mathbb{N} \setminus A$ is finite.

Proof. The sequence $d_k = \gcd\{a_1, \dots, a_k\} \geq 1$ is nonincreasing and $\in \mathbb{N}$, so

$$\underbrace{\gcd(A)}_{=: \lim_{k\to\infty} d_k} = \gcd\{a_1, \dots, a_k\} \text{ for some } k.$$

Moreover, $1 = \sum_{i=1}^{k} n_i a_i$ for some $n_i \in \mathbb{Z}$.

Separate positive and negative n_i's

$\Rightarrow$ $1 = P - N$, where $P, N \in A$.

Let $n \geq N(N-1)$. Then $n = qN + r$, where $0 \leq r \leq N-1$ and $q \geq N-1$.

Now

$$n = qN + r = qN + r(P-N)$$
$$= (q-r)N + rP \in A.$$

$\square$

• Corollary. Irreducible & aperiodic & finite

$\Rightarrow$ $\exists k \geq 1 : (T^k)_{ii} \geq 1 \quad \forall i$.

ACTUALLY $\exists N \geq 1 : k \geq N \Rightarrow (T^k)_{ii} \geq 1 \quad \forall i$. The set $\{k \geq 1 : (T^k)_{ii} \geq 1\}$ satisfies the assumptions in the Lemma, for $\forall i$. $\square$

• Corollary. ——— " ——— (SAME AS ABOVE)

$\Rightarrow$ $\exists k \geq 1 : (T^k)_{ij} \geq 1 \quad \forall i, j$.

AGAIN $\exists N \geq 1 : k \geq N \Rightarrow$

Proof.

Consider two states i and j. Let $M \in \mathbb{N}$ be such that (FROM PREVIOUS COROLLARY)

$$n \geq M \implies (T^n)_{ii} \geq 1 \quad \forall i,$$

and let $m = m(i,j)$ be such that

$$(T^m)_{ij} \geq 1.$$

Let $\tilde{N} = \tilde{N}(i,j) = M + m(i,j)$. Then

$$(T^{\tilde{N}})_{ij} \geq (T^M)_{ii}\,(T^m)_{ij} \geq 1,$$

and further

$$n \geq \tilde{N} \implies (T^n)_{ij} \geq 1.$$

Choose

$$N = \max\{\tilde{N}(i,j) : i,j \text{ states}\}. \qquad \square$$

BACK TO MARKOV CHAINS. (Random walks on state graphs.)

- <u>Proposition</u>.

a) $P \in \mathbb{R}^{n\times n}$ Markov $\implies$ P^k Markov $\forall k$.

b) $\bar{x}$ prob. vector & P Markov

$\implies$ $\bar{x}P$ prob. vector. ($\bar{x}$ ROW VECTOR.)

<u>Proof</u>. Exrc. 3 / Prob. 2. $\square$

- Denote

$$M^{n\times n} = \{P \in \mathbb{R}^{n\times n} \mid P \text{ Markov}\},$$

$$S_n = \{\bar{x} \in \mathbb{R}^n \mid \bar{x} \text{ prob. vector}\}.$$

Further, let $\overline{x}_0 \in S_n$ be such that

$$(\overline{x}_0)_i = \begin{cases} 1 & \text{for some (one) } i \in \{1, \dots, n\} \\ 0 & \text{otherwise.} \end{cases}$$

- Define the <u>Markov process</u> with <u>initial state</u> $\overline{x}_0$ and <u>transition probability matrix</u> P as a sequence

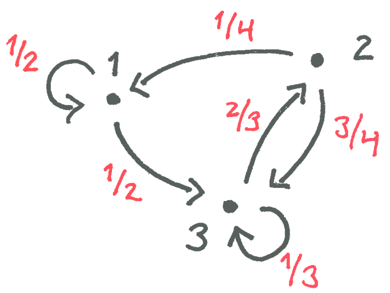
$$(\overline{x}_k)_{k=0}^{\infty} \subset S_n ,$$

where $\overline{x}_k = \overline{x}_0 P^k$.

- <u>Interpretation.</u> On the weighted state graph determined by P, start a <u>random walk</u> from the state determined by $\overline{x}_0$. Then $(\overline{x}_k)_i$ is the probability for being at the state i after k steps.

($(P^k)_{ij}$: Probability of $i \to \dots \to j$ with k steps.)

- <u>Example.</u>



$$P = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 1/4 & 0 & 3/4 \\ 0 & 2/3 & 1/3 \end{pmatrix}$$

Here, if we start from state 3, the probabilities after one step are

$$\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 1/2 \\ 1/4 & 0 & 3/4 \\ 0 & 2/3 & 1/3 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 0 & 2/3 & 1/3 \end{pmatrix}}}$$

and after two steps

$$\bar{x}_0 P^2 = \bar{x}_0 P P = \bar{x}_1 P$$

$$= \begin{pmatrix} 0 & 2/3 & 1/3 \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 1/2 \\ 1/4 & 0 & 3/4 \\ 0 & 2/3 & 1/3 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1/6 & 2/9 & 11/18 \end{pmatrix}}}.$$

- <u>Remark</u>.
  The "initial state" can be any $\bar{x}_0 \in S_n$ (each state has its own initial probability).

- A Markov matrix $P \in \mathbb{R}^{n \times n}$ is <u>irreducible</u> and <u>aperiodic</u> if $\exists N \in \mathbb{N}$:

  $$k \geq N \implies (P^k)_{ij} > 0 \quad \forall i,j \in \{1,\dots,n\}.$$

  This corresponds to an irred. & aper. state graph with non-zero weights.

- A prob. vector $\bar{x} \in S_n$ is a <u>stationary distribution</u> for the Markov process determined by $P \in \mathbb{R}^{n \times n}$ if

  $$\bar{x} P = \bar{x}. \qquad \left(\text{Then also } \bar{x} P^k = \bar{x} \ \forall k.\right)$$

• **Theorem.** (Perron 1907, Frobenius 1912)

Let $P \in \mathbb{R}^{n \times n}$ (Markov) be irred. & aper.
Then

a) P has a unique stationary distribution $\bar{x} \in S_n$.

b) The Markov process converges to the stationary distribution, no matter what the initial state is:

$$\bar{x}_0 P^k \xrightarrow[k \to \infty]{} \bar{x} \qquad \forall \bar{x}_0 \in S_n .$$

c) The stationary distribution is the left eigenvector of P (i.e. the eigenvector of P^T) corresponding to the eigenvalue 1.

About the proof.

The item c) is just the definitions:

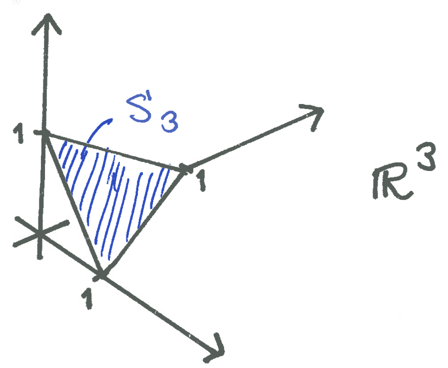
$$\bar{x} P = 1 \bar{x} \iff (\bar{x} P)^T = 1 \bar{x}^T \iff P^T \bar{x}^T = 1 \bar{x}^T .$$

$\bar{x}$ ROW VECT.
$\bar{x}^T$ COL. VECT.

About b) & a):

The prob. vectors S_n form the <u>$(n-1)$-dimensional unit simplex</u> ($\approx$ tetrahedron) in $\mathbb{R}^n$.

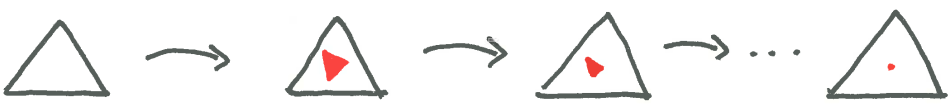
E.g. $n = 3$:



Our Markov process can be thought of as a sequence of points in the simplex.
Irreducibility & aperiodicity $\Longleftrightarrow$ contraction:

(After N steps the point is in the red triangle, independent where it started.)

As the process goes on:



$\square$

Wed 29 Apr

- The stationary distribution answers the question: "what are the visiting frequencies of the states in the long run?"
- Example. With $P = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 1/4 & 0 & 3/4 \\ 0 & 2/3 & 1/3 \end{pmatrix}$,

~~one~~ an eigenvector with eigenvalue 1 is $(1, 2, 3)$. The stationary distribution is the corresponding prob. vector, i.e.

$$\bar{x} = \frac{1}{6}(1,2,3) = (1/6, 1/3, 1/2)$$

$$\approx \underline{\underline{(17\%, 33\%, 50\%)}}$$

E.G. STATE 2 IS VISITED 33% OF THE TIME.

V The (actual) juggling Markov chain: the finite & uniform case

IV SHOULD HAVE BEEN: GENERAL FINITE MARKOV THEORY.

- The (b,h) state graph is irreducible (clearly) and aperiodic (ground state loop), so a unique stationary distribution always exists, no matter how we weigh the throws.
- Problem: hard to form the Markov matrix & even harder to determine its eigenvectors! NUMERICALLY NOT TOO HARD, BUT WE WANT COMBINATORIAL FORMULAS.
- In a (b,h) state B containing 0, there are $h-b+1$ possible throws. (If $0 \notin B$, there is only one - the zero-throw.) The simplest case (mathematically) is to assign the probability $\frac{1}{h-b+1}$ to each of these. (And probability one for the zero-throws.) "UNIFORM THROWS!"
- For a state $B \in \underset{\text{(b,h) GRAPH}}{G(b,h)}$ and $x \in B$, define

$$v_B(x) = \left| \{x, \dots, h-1\} \setminus B \right| ,$$

i.e. the vacant slots (or holes) above x.

Then we have

- Theorem. (Warrington 2005)
In random juggling with uniform throws, the stationary probability for a state $B \in G(b,h)$ is

$$\frac{\prod'_{x\in B} (1+\vartheta_B(x))}{S(h+1,\, h-b+1)} ,$$

where $S(m,k)$ is the Stirling number of the second kind, defined by the recursion

$$S(m+1,k) = S(m,k-1) + k\,S(m,k)$$

with the initial conditions

$$S(0,0)=1 \quad \text{and} \quad S(m,k)=0 \quad \text{for } k<0 \text{ or } k>m.$$

- We now set out to prove this.
First some warm-up: E.G. $S(1,1) = \underbrace{S(0,0)}_{=1} + 1\,\underbrace{S(0,1)}_{=0} = 1.$

- Proposition. The number $S(m,k)$ counts the number of ways of partitioning an m-element set into k blocks (irrespective of order).

For example, $S(4,2)=7$ as we can partition $\{A,B,C,D\}$ into two parts in seven ways:

A | BCD , B | ACD , C | ABD , D | ABC,

AB | CD , AC | BD , AD | BC .

Proof. HERE WE PROVE: IF THE # OF PARTITIONS IS DENOTED BY $S(n,k)$, THEN $S(n,k)$ SATISFIES THE STIRLING RECURSION.

A partition of $n+1$ elements into k blocks either has the ~~element~~ $(n+1)$-th element as a separate block or not.

The # of partitions that have it separate is $S(n, k-1)$.

In the other case, the $(n+1)$-th element belongs to a block containing some other elements as well. We partition the other n elements into k blocks ($S(n,k)$ ways) and insert the $(n+1)$-th element into one of them (k choices).

In total, there are

$$S(n, k-1) + k\,S(n,k)$$

ways of partitioning $n+1$ elements into k blocks, and

$$\underbrace{S(1,1) = 1}_{\text{CLEAR}}, \quad \underbrace{S(1,0) = S(0,1) = 0}_{\text{WE ACCEPT THESE. } S(0,0)=1 \text{ FOLLOWS FROM THE RECURSION.}}$$

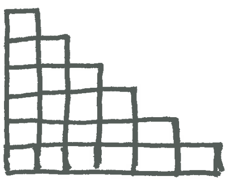
$\square$

- Strategy for proving the Theorem:
  - Define an extended process (states = rook placements on staircases)
  - Prove that the extended process is doubly stochastic (so its stationary distribution is uniform) (EXERCISES)
  - Project the extended process into the original one.

Formula explained: there are

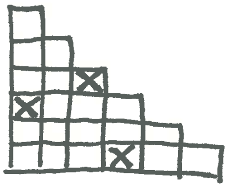
$\prod_{x\in B}(1+\vartheta_B(x))$ ways to ~~expand~~ extend a state B, and there are $S(h+1, h-b+1)$ extended states in total. (EXERCISES)

- A staircase Ferrers board of size h is a diagram like the following ($h=6$):



$1+\dots+h = \binom{h+1}{2}$ EMPTY SLOTS.

- A non-attacking rook placement with b rooks on a h-staircase is a diagram like the following ($h=6$, $b=3$):



AT MOST ONE ROOK PER ROW / COLUMN ALLOWED.

• The extended (b,h) juggling process

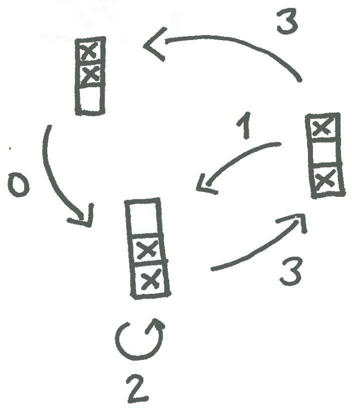
has

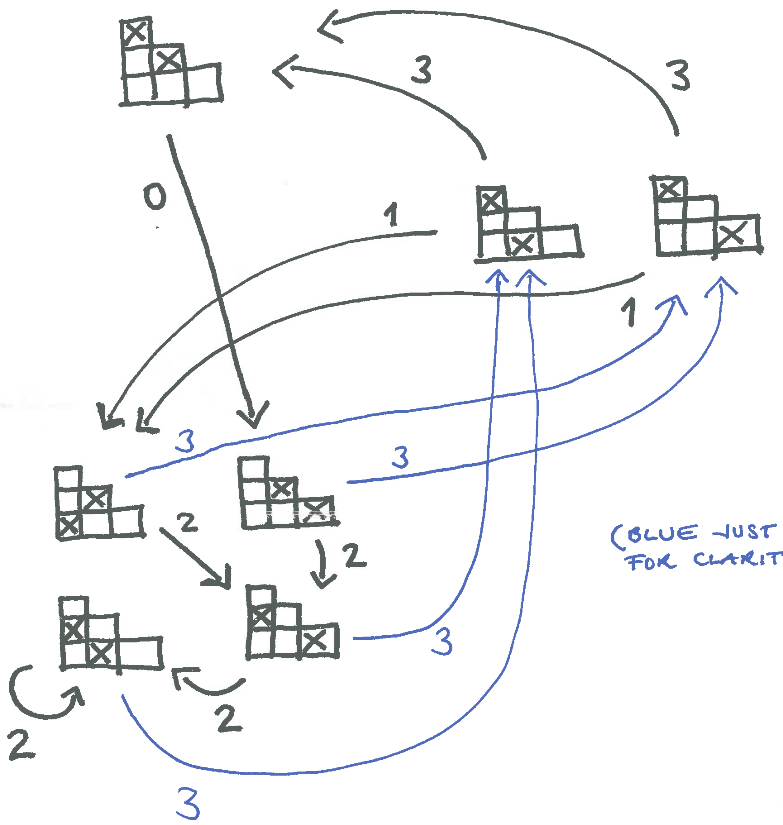
• states: b non-attacking rooks ("PARTICLES") on a staircase Ferrers board of size h

EXERCISE: THERE ARE $S(h+1,\ h-b+1)$ OF THEM!

• dynamics: the particles drift ~~down~~ diagonally southeast and are thrown from the bottom row to the leftmost column.

• Example. The $(2,3)$ graph and its extended counterpart:





(BLUE JUST FOR CLARITY.)

IDEA: