- <u>Lemma.</u> There are $S(h+1, h-b+1)$ extended (b,h) states.

<u>Proof.</u>

Denote the # of extended (b,h) states by $E(b,h)$.

An ext. $(b,h+1)$ state is obtained by either placing all the b rooks in other than the bottom row or by placing one of the rooks in the bottom row.

In the first case there are $E(b,h)$ ways, in the second case there are $\underbrace{(h-b+2)}_{=(h+1)-(b-1)} E(b-1,h)$ ways. Thus

$$E(b,h+1) = E(b,h) + (h-b+2)E(b-1,h),$$

i.e. (EXERCISE)

$$E(b,h) = S(h+1, h-b+1).$$

$\square$

- <u>Lemma.</u> A state $B \in G(b,h)$ can be extended in

$$\prod_{x\in B}\left(1+v_B(x)\right) = \prod_{x\in B}\left(1+\left|\{x,\dots,h-1\}\setminus B\right|\right)$$

ways.

Proof.

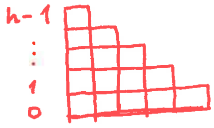
Induction on b. If $b = 1$ AND $B = \{x\} \subset \{0, \dots, h-1\}$, there are $(h-1) - x + 1$ ways to extend B.

If $b \geq 2$, consider $x = \min B$. By the induction assumption, there are

$$\prod_{y \in B \setminus \{x\}} \left(1 + v_B(y)\right)$$

ways to extend $B \setminus \{x\} \subset \{0, \dots, h-1\}$,

and there are



h ROWS NUMBERED FROM 0 TO $h-1$.

$$h - x - (b-1)$$

ways of placing a non-attacking rook to row x after $b-1$ rooks have already been placed to higher rows.

Because

$$|\{x, \dots, h-1\} \setminus B| = h - x - b,$$

I.E. $1 + |\{x, \dots, h-1\} \setminus B| = h - x - (b-1)$

the claim follows.

$\square$

• Lemma. The extended (b,h) process is doubly stochastic.

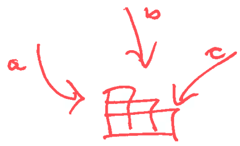
Proof.

Let P be the Markov matrix of the ext. (b,h) process.

Columns of P sum to 1

$\iff$

The weights of the arrows entering each extended state sum to 1.

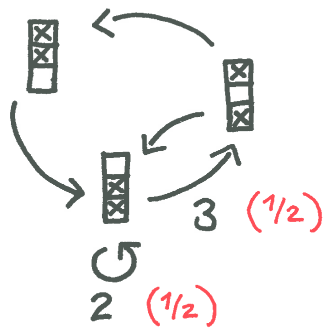


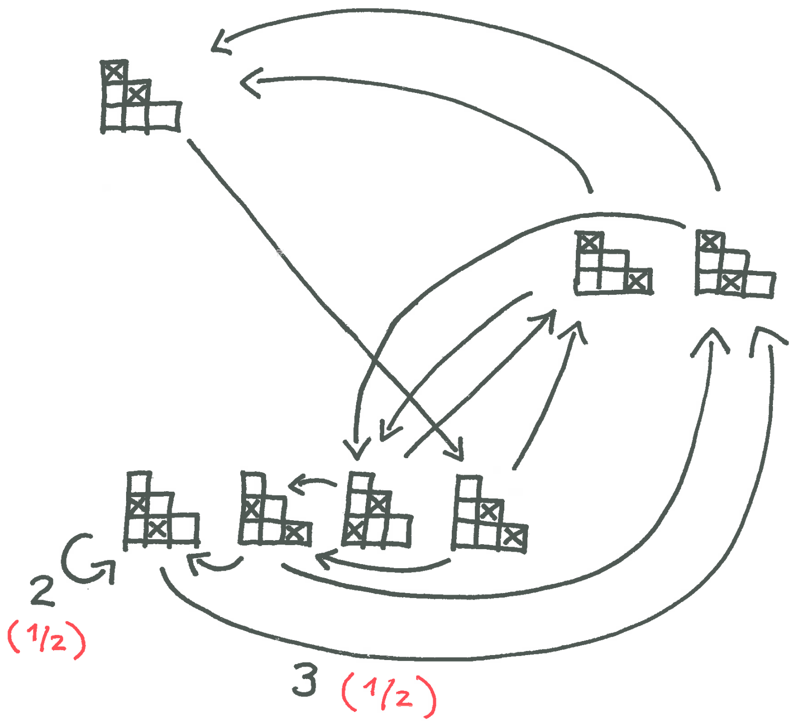
$a + b + c = 1$?

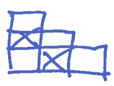
ADDENDUM TO PAGE 33 :

- What do we mean by the extended (b, h) process ?

- Redraw picture on p. 31 :





- Note : Each extended state has the same arrows leaving it as has the corresponding non-extended state. (Example :  and  - probability weights in red.)

- We can think of a random walk running in parallel on both graphs :
If started on the extended side, the non-extended counterpart is unique.
If started on the non-extended side, choose any extension of the starting state.
Thereafter unique extended counterpart.

Consider an ext. state $\hat{B}$. If its left column is empty, then there is only one inarrow (zero-throw, weight 1). Otherwise $\hat{B}$ has $h-b+1$ inarrows(*), each of which has weigth $(h-b+1)^{-1}$.

(*): THINK WITH A SIMPLE EXAMPLE. (E.G. SEE PIC. P. 31.)

□

• **Lemma.** The extended process is irreducible and aperiodic.

Proof: exercise.

• **Theorem.** The stationary probability of a state $B \in G(b,h)$ in the uniform process is

$$\frac{\prod_{x \in B} (1 + \vartheta_B(x))}{S(h+1,\, h-b+1)} .$$

"THE EXTENDED PROCESS IS IRREDUCIBLE AND APERIODIC"

Proof.

• Ext. proc. irred. & aper.

$\Rightarrow$ unique stationary ~~[illegible]~~ distribution

• Ext. proc. doubly stoch.

$\Rightarrow$ stat. dbution uniform; each ext. state has probability $1/S(h+1,\, h-b+1)$

• Imagine the extended and non-extended processes running in parallel.

Each visit to a non-extended state B corresponds (in the long run) to $\prod_{x \in B} (1 + v_B(x))$ visits to its extended states, each having probability $1/S(h+1, h-b+1)$.

$\square$

VI The countably infinite case.

- Consider $G(b, \infty)$, the b-particle siteswap state graph with no bound on the throw height.

- State $B \in G(b, \infty)$: any $B \subset \mathbb{Z}_+ = \{x \in \mathbb{Z} \mid x \geq 0\}$ with $|B| = b$.

- Dynamics: exactly as in the $h < \infty$ case (p. 7; replace $\{0, \ldots, h-1\}$ by $\mathbb{Z}_+$).

↑ THESE DEFINE THE (b, ∞) STATE GRAPH.

- Our aim:

  1) Concentrate on the $|B| = 2$ case & introduce concepts from countable Markov chains,

  2) Replace "uniform throws" (finite) by "geometric throws" (countable)

and derive an explicit formula for the stationary distribution.

- A <u>juggling process</u> (= a walk on the state graph) is a sequence of states $(B_n)_{n=0}^{\infty}$ such that B_{n+1} is a follower of B_n, for each n.

- <u>Example</u>. With $b=2$, a state has the form $B=\{i,j\}$. If $0 \notin B$, its only follower (successor) is $\{i-1, j-1\}$ $(= B-1)$. If $0 \in B$, it has countably many successors: $(B^*-1) \cup \{x\}$ for any $x \in (B^*-1)^c$, where $B^* = B \setminus \{0\}$.

(GENERAL DYNAMICS REPEATED.)

- <u>Example</u> (continuation).
What are the <u>predecessors</u> of $B=\{i,j\}$? I.e. if $B_{n+1}=\{i,j\}$, what can B_n be? Answer:

$$\underbrace{\{i+1, j+1\}}_{\text{JUGGLER WAITED}}, \quad \underbrace{\{0, i+1\} \text{ or } \{0, j+1\}}_{\text{JUGGLER THREW ONE OF THE PARTICLES}}.$$

- Introduce random throws: $(b=2)$

Let $h^j(i) \in [0,1]$ denote the probability of $\{0, j+1\} \longrightarrow \{i, j\}$, i.e.

$$\boxed{h^j(i) = \text{probability of throwing to "height } i\text{" while the other particle fell from } j+1 \text{ to } j.}$$

- Then the <u>one-step equation</u> reads

$$\mathbb{P}(B_{n+1} = \{i, j\})$$
$$= \mathbb{P}(B_n = \{i+1, j+1\})$$
$$+ \mathbb{P}(B_n = \{0, j+1\})\, h^j(i)$$
$$+ \mathbb{P}(B_n = \{0, i+1\})\, h^i(j),$$

ELEMENTARY. COMPARE WITH PREDECESSORS ON PREVIOUS PAGE.

where $\mathbb{P}(B_n = \{i,j\})$ denotes the probability that the n-th state is $\{i,j\}$.

- The one-step equation, along with a starting state B_0 (or a known distribution $\mathbb{P}(B_0 = \{i,j\})$), determines the random juggling process.

- A <u>probability distribution</u> on a countable set Ω is a function $\mu : \Omega \to [0,1]$ such that $\sum_{x \in \Omega} \mu(x) = 1$.

- <u>Example</u>.

$= \frac{1}{2} \cdot \frac{1}{2^x} = (1-\frac{1}{2}) \cdot \frac{1}{2^x}$

$\mu(x) = \frac{1}{2^{x+1}}$ is a (geometric) prob. distribution on $\mathbb{Z}_+$, because $\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} = 1$.

- <u>Example</u>.

Let $0 < a < 1$. Then

$$h^j(i) = \begin{cases} (1-a)\, a^{i-1}, & i > j \\ (1-a)\, a^i, & i < j \\ 0, & i = j \end{cases}$$

defines random throws $(b = 2)$, because $\forall j \geq 0$:

$$\sum_{i=0}^{\infty} h^j(i) = (1-a)\Big(\underbrace{\sum_{i=0}^{j-1} a^i}_{\text{ZERO IF } j=0} + \sum_{i=j+1}^{\infty} a^{i-1} \Big)$$

$$= (1-a) \underbrace{\sum_{i=0}^{\infty} a^i}_{= \frac{1}{1-a}} = 1 .$$

Call this the <u>a-geometric throw distribution</u>.

- <u>Example.</u>

  Let $0 < a < 1$. Then

  $\frac{1}{a}(1-a)(1-a^2)$

  $$\mu(\{i,j\}) = \cancel{\tfrac{1}{2}(1-a)^2} a^{i+j}$$

  defines a probability distribution on the set of $(2,\infty)$ states

  $$\{\{i,j\} \mid i,j \in \mathbb{Z}_+, i \neq j\}.$$ EXERCISE.

  Call this the <u>a-geometric height distribution</u>.

- Rewrite the one-step equation $(b=2)$ as

  $$\mu_{n+1}(\{i,j\})$$
  $$= \mu_n(\{i+1, j+1\}) + \mu_n(\{0, j+1\}) h^j(i) + \mu_n(\{0, i+1\}) h^i(j),$$

  where $\mu_n(A) = \mathbb{P}(B_n = A)$.

- Given the throw distribution $h^j(i)$ (actually a family of distributions), a distribution μ on $(2,\infty)$ states is a <u>stationary distribution</u> if $\mu = \mu_n \; \forall n$, i.e. if

$$\mu(\{i,j\}) = \mu(\{i+1,j+1\}) + \mu(\{0,j+1\})h^j(i) + \mu(\{0,i+1\})h^i(j).$$

Call this the <u>balance equation</u> (or equilibrium equation, or stationarity equation) of the 2-particle process.

FAMILIAR INTERPRETATION: $\mu(B)$ IS THE LONG-TERM VISITING PROBABILITY FOR B.

- <u>Two questions</u>.

1) Which throw distributions $h^j(i)$ admit stationary height distributions $\mu(\{i,j\})$?

WE WILL FIND A GENERAL CONDITION.

2) Can we solve the balance equation explicitly in some special cases? YES.

- <u>Example</u>.

Start from the ground state $B_0 = \{0,1\}$ and let

$$h^j(i) = \begin{cases} 1, & i = 2(j+1) \\ 0, & \text{otherwise}. \end{cases}$$

Then $B_1 = \{0,2\}$, $B_2 = \{1,4\}$, $B_3 = \{0,3\}$, $B_4 = \{2,6\}$ etc.

In particular:

$$\mu_n(B) = \begin{cases} 1 & \text{for a certain } B \\ 0 & \text{for others}, \end{cases}$$

and <u>no state is visited twice</u>, so there cannot exist a stationary distribution: The long-term visit probability (frequency) for each state is zero.

- <u>In general</u>:

Let G be (the states of) a countable state graph. A <u>probability transition operator</u> is a function

$$\mathbb{P} : G \times G \longrightarrow [0,1] \quad \text{such that}$$

$$\forall A \in G : \sum_{B \in G} \mathbb{P}(A,B) = 1 .$$

(WE ALSO ASSUME THAT $\mathbb{P}(A,B) > 0$ ONLY WHEN B FOLLOWS A IN THE GRAPH.)

Start from a state $B_0 \in G$, and denote $\mu_n(B) = \mathbb{P}(B_n = B)$. Then

$$\mu_{n+1}(B) = \sum_{A \in G} \mu_n(A)\,\mathbb{P}(A,B), \qquad \text{(1-STEP)}$$

and a probability distribution μ on G is <u>stationary</u> for the process determined by the operator $\mathbb{P}$ if

$$\mu(B) = \sum_{A \in G} \mu(A)\, P(A,B)$$

for each $B \in G$.

↖ "μ IS AN EIGENVECTOR FOR P WITH EIGENVALUE 1".

- In general, solving stationary equations is hard or impossible.

- Example.

  The stationary equation for the one-particle process reads

$$\mu(i) = \mu(i+1) + \mu(0)\, h(i),$$

and this can be solved generally (give μ in terms of h). EXERCISE.