- Note. In the previous example one needs
$$\underbrace{\sum_{i=0}^{\infty} i\,h(i)}_{\text{EXPECTED (= AVERAGE) THROW HEIGHT}} < \infty .$$

- Irreducibility & aperiodicity are defined as in the finite case: (THEY ALSO MEAN THE SAME.)

the process (def'd by $\mathbb{P}$) is irreducible if
$$\forall i,j \;\; \exists k = k(i,j) : \quad \mathbb{P}^k(i,j) > 0$$
and aperiodic if $\forall i$:
$$\gcd\left\{k \geq 1 : \mathbb{P}^k(i,i) > 0\right\} = 1 .$$
(BETTER WITH A, B THAN i, j.)

- However, in the countable case irreducibility & aperiodicity do not guarantee the existence of an equilibrium.

- Consider a (RANDOM JUGGLING) process defined by $B_0 = A$ and $\mathbb{P}$. Define the <u>first return time to A</u> as
$$\tau_A = \min\left\{n \geq 1 : B_n = A\right\},$$

and denote the probability of returning to A (at some point) by

$$r_A = \mathbb{P}(\tau_A < \infty).$$

- The ~~process~~ state A is <u>recurrent</u> if $r_A = 1$. Otherwise A is <u>transient</u>.

- <u>Note.</u> The probability of returning to A m times is r_A^m. Thus, if $r_A < 1$, then $r_A^m \xrightarrow[m\to\infty]{} 0$, i.e. the process does not visit A infinitely many times. (WITH POSITIVE PROBABILITY. ON THE OTHER HAND, $r_A = 1 \Rightarrow r_A^m = 1 \;\; \forall m$.)

- <u>Proposition.</u> If the state graph is irreducible and if one state is recurrent, then each state is recurrent.

"<u>Proof</u>." Because $A \longrightarrow B \longrightarrow A$ (IN $<\infty$ STEPS) with positive probability and because A is visited infinitely often with probability one, B is visited infinitely often with probability one. $\square$

- <u>Note</u>. In the one-particle process $r_A = 1$ for each A, i.e. the process is ALWAYS recurrent. However, if

$$\sum_{i=0}^{\infty} i\,h(i) = \infty ,$$

E.G. IF $h(i) = \frac{1}{i^2}$

then the <u>expected return time</u> from a state to itself is infinite!

(A STATIONARY DISTRIBUTION CANNOT BE DEFINED.)

- In general, it may happen that

$$\mathbb{P}(\tau_A < \infty) = 1 ,$$

but

$$\mathbb{E}(\tau_A) = \infty .$$

Such a process is called <u>null recurrent</u>.

- If $\mathbb{E}(\tau_A) < \infty$ for each A (sufficient: irreducibility & one A), then the process is <u>positive recurrent</u>.

- <u>Fact</u>.

a) A positive recurrent process has a stationary distribution.

b) A positive recurrent, irreducible & aperiodic process converges to a unique stationary distribution, independent of the starting state.

Proof: see Durrett, Sec. 1.7.

$\square$

PHILOSOPHY OF a): IF $\mathbb{E}(\tau_A) < \infty$, THEN $\mu(A) = \frac{1}{\mathbb{E}(\tau_A)}$.

- The basic tool for proving pos. recurrence is the

TOOL

<u>Foster-Lyapunov-theorem</u>.

Let $\mathbb{P}$ be a probability transition operator on a countable, irreducible and aperiodic state graph G.

If there exists a function $f: G \to [0,\infty)$ and a finite set $K \subset G$ such that

$\forall A \notin K$:

$$Df(A) := \sum_{B \in G} f(B)\mathbb{P}(A,B) - f(A) < -\varepsilon$$

for some $\varepsilon > 0$, then the process is positive recurrent.

<u>Philosophy</u>:

- $$Df(A) = \mathbb{E}\big(f(B_1) - f(B_0) \mid B_0 = A\big)$$

is the expected ONE-STEP change in the "energy" of a state

$f(A)$

- If $Df < -\varepsilon$ in one step, then

THE EXPECTED CHANGE IN f

the expected long-run change in f would be $-\infty$ (impossible because $f \geq 0$).

- Therefore the process must return to K again and again. $\square$

A GENERAL PROBLEM: FIND A SUITABLE f. (CALLED A LYAPUNOV FUNCTION.)

- Theorem. (Two particles.)

Let the throw distribution $h^j(i)$ be such that

$$\sup_j \sum_{i=m}^{\infty} i\, h^j(i) \xrightarrow[m \to \infty]{} 0 .$$

Then the process is positive recurrent.

Proof.

Choose m such that

$$\sup_j \sum_{i=m}^{\infty} i\, h^j(i) \leq \frac{1}{2} ,$$

let $K = \{ \{i,j\} \mid i,j \leq m \}$,

and let $f(\{i,j\}) = \max\{i,j\}$.

Two cases:

1) If $i, j > 0$, then

$$Df(\{i,j\}) = \max\{i-1, j-1\} - \max\{i,j\}$$
$$= -1 .$$

2) $Df(\{0, j+1\})$

$$= \sum_{i=0}^{\infty} \max\{i,j\}\, h^j(i) - \max\{0, j+1\}$$

$$= \sum_{i=0}^{j} \max\{i,j\} h^j(i) + \sum_{i=j+1}^{\infty} \max\{i,j\} h^j(i) - \max\{0, j+1\}$$

$$= j \underbrace{\sum_{i=0}^{j} h^j(i)}_{\leq \sum_{i=0}^{\infty} h^j(i) = 1} + \sum_{i=j+1}^{\infty} i\, h^j(i) - (j+1)$$

$$\leq -1 + \underbrace{\sum_{i=j+1}^{\infty} i\, h^j(i)}_{\leq \frac{1}{2}} \leq -\frac{1}{2}$$

when $j \geq m$, i.e. when $\{0, j+1\} \notin K$.

$\square$

- Recall the equilibrium equation for two particles:

$$\mu(\{i,j\}) = \mu(\{i+1, j+1\})$$
$$+ \mu(\{0, j+1\})h^j(i)$$
$$+ \mu(\{0, i+1\})h^i(j).$$

- <u>Theorem.</u>

Let

$$h^j(i) = \begin{cases} (1-a)a^i, & i<j \\ (1-a)a^{i-1}, & i>j \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\mu(\{i,j\}) = c\,a^{i+j},$$

where

$$c^{-1} = \frac{a}{1-a} \cdot \frac{1}{1-a^2}.$$

<u>Proof.</u> ASSUME $i<j$.

Plug into the equilibrium equation:

$$\cancel{c}\,a^{i+j} = \cancel{c}\,a^{i+j+2} + \cancel{c}\,a^{j+1}\cdot(1-a)a^i$$
$$+ \cancel{c}\,a^{i+1}\cdot(1-a)a^{j-1}$$

$$= a^{i+j}\left(\underbrace{a^2 + (1-a)a + (1-a)}_{=a^2+a-a^2+1-a \;=\; 1}\right)$$

$$= a^{i+j} \quad \text{OK !} \qquad \text{SIMILAR IF } i > j.$$

The constant satisfies (BECAUSE $\sum_{\{i,j\}} c\mu(\{i,j\}) = 1$)

$$\frac{1}{c} = \sum_{\{i,j\}} a^{i+j} = \sum_{i=1}^{\infty} \sum_{j=0}^{i-1} a^{i+j}$$

$$= \sum_{i=1}^{\infty} a^i \sum_{j=0}^{i-1} a^j = \sum_{i=1}^{\infty} a^i \cdot \frac{1-a^i}{1-a}$$

$$= \frac{1}{1-a}\left(\sum_{i=1}^{\infty} a^i - \sum_{i=1}^{\infty} (a^2)^i\right)$$

$$= \frac{1}{1-a}\left(\frac{(1+a)a}{1-a} - \frac{a^2}{1-a^2}\right)$$

$$= \frac{1}{1-a}\left(\frac{a + \cancel{a^2} - \cancel{a^2}}{1-a^2}\right) = \underline{\underline{\frac{a}{(1-a)(1-a^2)}}}.$$

CLOSING REMARKS.

- The unbounded case is similar for n particles. (LESKELÄ - VARPANEN 2012)
- Both the bounded and unbounded cases can be solved for general

$$h(i) = \mathbb{P}(\text{throw to } i\text{-th available height}),$$

and in the unbounded case

$$\sum_{i=1}^{\infty} i\,h(i) < \infty$$

is necessary & sufficient for existence. (AYYER ET AL. MULTIVARIATE JUGGLING PROBABILITIES, 2015)

- <u>Open problem</u>. In the ∞ case, ~~#~~ let $h(i) = \frac{1}{i^{1+\alpha}}$, where $0<\alpha<1$.

Then $\sum_{i=1}^{\infty} i\,h(i) = \sum_{i=1}^{\infty} \frac{1}{i^{\alpha}} = \infty$,

so the process is either null recurrent or transient.

<u>Conjecture</u> (Ayyer et al. 2015):

null recurrent $\iff$ $n(1-\alpha) \leq 1$.

OF PARTICLES